

STRICT SINGULARITY OF A VOLTERRA-TYPE INTEGRAL OPERATOR ON H^p

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ABSTRACT. We prove that a Volterra-type integral operator

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta, \quad z \in \mathbb{D},$$

defined on Hardy spaces H^p , $1 \leq p < \infty$, fixes an isomorphic copy of ℓ^p , if the operator T_g is not compact. In particular, this shows that the strict singularity of the operator T_g coincides with the compactness of the operator T_g on spaces H^p . As a consequence, we obtain a new proof for the equivalence of the compactness and the weak compactness of the operator T_g on H^1 .

1. INTRODUCTION

Let g be a fixed analytic function in the open unit disc \mathbb{D} of the complex plane \mathbb{C} . We consider a linear integral operator T_g defined formally for analytic functions f in \mathbb{D} by

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta, \quad z \in \mathbb{D}.$$

Ch. Pommerenke was the first author to consider the boundedness of the operator T_g on Hardy space H^2 and he characterized it in [12] in a connection to exponentials of $BMOA$ functions. A systematic study of the operator T_g was initiated by A. Aleman and A. G. Siskakis in [4], where they stated the boundedness and compactness characterization of T_g on Hardy spaces H^p , $1 \leq p < \infty$. Namely, they observed that T_g is bounded (compact) if and only if $g \in BMOA$ ($g \in VMOA$). The same boundedness characterization of the operator T_g on H^p , $0 < p < 1$, spaces was obtained by Aleman and J. Cima in [2]. Many properties of the operator T_g have been studied by several authors later on and they are well known in most spaces of analytic functions, see also surveys [1] and [13].

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However, one operator theoretically interesting property, the strict singularity, has not been considered in the case of T_g . A bounded operator $S: X \rightarrow Y$ between Banach spaces is strictly singular if its restriction to any infinite-dimensional closed subspace is not an isomorphism onto its image. This notion generalizes the concept of compact operators and it was introduced by T. Kato in [7]. Canonical examples of strictly singular non-compact operators are inclusion mappings $i_{p,q}: \ell^p \hookrightarrow \ell^q$, where $1 \leq p < q < \infty$. There also exist strictly singular non-compact operators on H^p spaces for $1 \leq p < \infty$, $p \neq 2$.

The aim of this note is to show that a non-compact operator T_g defined on Hardy spaces H^p , $1 \leq p < \infty$, fixes an isomorphic copy of ℓ^p . In particular, this implies that the operator T_g is strictly singular on H^p if and only if it is compact. Moreover, this gives a new proof for the equivalence of compactness and weak compactness of T_g on Hardy space H^1 , see [8].

Our main result is the following theorem.

Theorem 1.1. *Let $g \in BMOA \setminus VMOA$ and $1 \leq p < \infty$. Then the operator*

$$T_g: H^p \rightarrow H^p$$

fixes an isomorphic copy of ℓ^p inside H^p . In particular, the operator T_g is not strictly singular, i.e. the class of strictly singular operators T_g coincides with the class of compact operators T_g .

We should point out that there is an interesting extrapolation result by Hernández, Semenov, and Tradacete in [6, Theorem 3.3]. It states that if an operator S is bounded on L^p and L^q for some $1 < p < q < \infty$ and strictly singular on L^r for some $p < r < q$, then it is compact on L^s for all $p < s < q$. If the corresponding statement for L^p spaces of complex-valued functions is true, then the equivalence of strict singularity and compactness of T_g on H^p for $1 < p < \infty$ follows immediately by using the Riesz projection: Recall that strictly singular operators form a two-sided (closed) ideal in the space $\mathcal{L}(L^p)$ of bounded operators on $L^p = L^p(\mathbb{T})$, where $\mathbb{T} = \partial\mathbb{D}$. Therefore the strict singularity of $T_g: H^p \rightarrow H^p$ implies that $T_g R: L^p \rightarrow L^p$ is strictly singular, where $R: L^p \rightarrow H^p$ is the Riesz projection and we have identified $T_g: H^p \rightarrow H^p$ with $T_g: H^p \rightarrow L^p$. Since the condition $g \in BMOA$ characterizes the boundedness of T_g on every H^q , $0 < q < \infty$, space and the Riesz projection is bounded on the scale $1 < q < \infty$, we get that $T_g R$ is bounded on every L^q , $1 < q < \infty$, space. Now assuming that the complex version of the interpolation result is valid, it follows that $T_g R$ is compact on L^p and consequently the restriction $T_g R|_{H^p} = T_g$ is compact on H^p .

However, Theorem 1.1 states more: a non-compact operator T_g on H^p fixes an isomorphic copy of ℓ^p and this is also true in the case $p = 1$.

Theorem 1.1 also gives a new proof for the equivalence of the compactness and the weak compactness of the operator T_g on H^1 : If $g \in BMOA \setminus VMOA$, i.e. the operator T_g is not compact, then by Theorem 1.1 the operator T_g fixes an infinite-dimensional subspace M , an isomorphic copy of ℓ^1 . The class of compact operators on ℓ^1 coincides with the class of weakly compact operators on ℓ^1 . As an isomorphism, the restriction $T_g|_M$ is not compact and hence it is not weakly compact. Therefore the operator T_g is not weakly compact.

2. PRELIMINARIES

In this section, we briefly remind a reader some common spaces of analytic functions that appear later and state a theorem of Aleman and Cima which we need in the proof our main result Theorem 1.1.

Let $H(\mathbb{D})$ be the algebra of analytic functions in \mathbb{D} . We define Hardy spaces

$$H^p = \left\{ f \in H(\mathbb{D}) : \|f\|_p = \left(\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p} < \infty \right\}.$$

Space $BMOA$ consists of functions $f \in H(\mathbb{D})$ with

$$\|f\|_* = \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_2 < \infty,$$

where $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$ is the Möbius automorphism of \mathbb{D} that interchanges the origin and the point $a \in \mathbb{D}$. Its closed subspace $VMOA$ consists of those $f \in H(\mathbb{D})$ with

$$\limsup_{|a| \rightarrow 1} \|f \circ \sigma_a - f(a)\|_2 = 0.$$

See e.g. [5] for more information on spaces $BMOA$ and $VMOA$. The Bloch space \mathcal{B} is the Banach space of functions $f \in H(\mathbb{D})$ s.t.

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

We use notation $A \lesssim B$ to indicate that $A \leq cB$ for some positive constant c whose value may change from one occurrence into another and which may depend on p . If $A \lesssim B$ and $B \lesssim A$, we say that the quantities A and B are equivalent and write $A \simeq B$.

Every $BMOA$ function f satisfies a reverse ‘‘Hölder’s inequality’’, which implies that for each $0 < p < \infty$ it holds that

$$\|f\|_* \simeq \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_p < \infty,$$

where the proportionality constants depend on p . Similarly, a function f is in $VMOA$ if and only if

$$\limsup_{|a| \rightarrow 1} \|f \circ \sigma_a - f(a)\|_p = 0.$$

The proof of Theorem 1.1 utilizes a result of Aleman and Cima [3, Theorem 3]. We state it here for convenience.

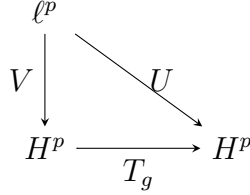
Theorem 2.1. *Let $p > 0$ and $g \in H^p$. For $a \in \mathbb{D}$, let $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$ and $f_a(z) = (1 - |a|^2)^{1/p}/(1 - \bar{a}z)^{2/p}$. Then for $0 < t < p/2$, there exists a constant $A_{p,t} > 0$ (depending only on p and t) such that*

$$\|g \circ \sigma_a - g(a)\|_t^t \leq A_{p,t} \|T_g f_a\|_p^t.$$

3. MAIN RESULT

Our goal is to show that a non-compact operator $T_g: H^p \rightarrow H^p$, $1 \leq p < \infty$, $g \in BMOA \setminus VMOA$, fixes an isomorphic copy of ℓ^p yielding that the compactness and strict singularity are equivalent for T_g on H^p . This is done by constructing bounded operators $V: \ell^p \rightarrow H^p$ and $U: \ell^p \rightarrow H^p$, where $V(\ell^p) = M$ is a closure of a linear span of suitably chosen test functions $f_{a_k} \in H^p$ and the operator U is an isomorphism onto its image $U(\ell^p) = T_g(M)$. Then it is straightforward to show that the restriction $T_g|_M: M \rightarrow T_g(M)$ is bounded from below by a positive constant and consequently an isomorphism, see Figure 1.

FIGURE 1. Operators U, V and T_g



The strategy for choosing the suitable test functions in Proposition 3.2 and Theorem 3.6 is similar to the one used by Laitila, Nieminen and Tylli in [10], where they utilized these test functions to show that a non-compact composition operator $C_\varphi: H^p \rightarrow H^p$, where $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ is analytic, fixes an isomorphic copy of ℓ^p .

Before proving our main result (Theorem 1.1), we need some preparations. We prove first a localization lemma for the standard test functions in H^p , $1 \leq p < \infty$, defined by

$$f_a(z) = \left[\frac{1 - |a|^2}{(1 - \bar{a}z)^2} \right]^{1/p}, \quad z \in \mathbb{D},$$

for each $a \in \mathbb{D}$.

Lemma 3.1. *Let $1 \leq p < \infty$, $\varepsilon > 0$ and $(a_k) \subset \mathbb{D}$ be a sequence s.t. $(|a_k|)$ is increasing and $a_k \rightarrow \omega \in \mathbb{T}$. Define*

$$A_\varepsilon = \{e^{i\theta} : |\theta - \arg(\omega)| < \varepsilon\}.$$

Then

- (i) $\lim_{k \rightarrow \infty} \int_{\mathbb{T} \setminus A_\varepsilon} |f_{a_k}|^p dm = 0.$
- (ii) *If k is fixed, then $\lim_{\varepsilon \rightarrow 0} \int_{A_\varepsilon} |f_{a_k}|^p dm = 0.$*

Proof. (i) Fix $\varepsilon > 0$. It holds that

$$|1 - \bar{a}_k \zeta| \gtrsim |1 - \bar{\omega} \zeta| \geq |\omega - \zeta| \gtrsim \varepsilon$$

for $\zeta \in \mathbb{T} \setminus A_\varepsilon$ and large enough k . Thus

$$|f_{a_k}(\zeta)|^p = \frac{1 - |a_k|^2}{|1 - \bar{a}_k \zeta|^2} \leq \frac{1 - |a_k|^2}{|\omega - \zeta|^2} \lesssim \frac{1 - |a_k|^2}{\varepsilon^2}$$

when $\zeta \in \mathbb{T} \setminus A_\varepsilon$, and it follows that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{T} \setminus A_\varepsilon} |f_{a_k}|^p dm = 0.$$

(ii) Fix k . It follows from the absolute continuity of a measure $B \mapsto \int_B |f_{a_k}|^p dm$ that $\int_{A_\varepsilon} |f_{a_k}|^p dm \rightarrow 0$ as $\varepsilon \rightarrow 0$. \square

Next, utilizing test functions $f_{a_k}, a_k \in \mathbb{D}$, for which $|a_k| \rightarrow 1$ sufficiently fast, we construct a bounded operator $V: \ell^p \rightarrow H^p$.

Proposition 3.2. *Let $1 \leq p < \infty$ and $(a_n) \subset \mathbb{D}$ be a sequence s.t. $(|a_n|)$ is increasing and $a_n \rightarrow \omega \in \mathbb{T}$. Then there exists a subsequence $(b_n) \subset (a_n)$ so that the mapping*

$$S: \ell^p \rightarrow H^p, S(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_n,$$

where $\alpha = (\alpha_n) \in \ell^p$ and $f_n = f_{b_n}$, is bounded. In particular, every mapping

$$V: \ell^p \rightarrow H^p, V(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_{c_n},$$

where $(c_n) \subset (b_n)$, is bounded.

Proof. For each $\varepsilon > 0$, we define a set $A_\varepsilon = \{e^{i\theta} : |\theta - \arg(\omega)| < \varepsilon\}$. Using the fact that $\|f_a\|_p = 1$ for all $a \in \mathbb{D}$ and Lemma 3.1, we choose

positive numbers ε_n with $\varepsilon_1 > \varepsilon_2 > \dots > 0$ and numbers $b_n \in (a_n)$ s.t. the following conditions hold

$$\begin{aligned} \text{(i)} \quad & \left(\int_{A_n} |f_j|^p dm \right)^{1/p} < 4^{-n}, \quad j = 1, \dots, n-1; \\ \text{(ii)} \quad & \left(\int_{\mathbb{T} \setminus A_n} |f_n|^p dm \right)^{1/p} < 4^{-n}; \\ \text{(iii)} \quad & \left(\int_{A_n} |f_n|^p dm \right)^{1/p} \leq \|f_n\|_p = 1 \end{aligned}$$

for every $n \in \mathbb{N}$, where $A_n = A_{\varepsilon_n}$.

Using conditions (i)-(iii), we show the upperbound $\|S\alpha\|_p \leq C\|\alpha\|_{\ell^p}$ for all $\alpha = (\alpha_j) \in \ell^p$, where $C > 0$ may depend on p .

$$\begin{aligned} \|S\alpha\|_p^p &= \int_{\mathbb{T}} \left| \sum_{j=1}^{\infty} \alpha_j f_j \right|^p dm = \sum_{n=1}^{\infty} \int_{A_n \setminus A_{n+1}} \left| \sum_{j=1}^{\infty} \alpha_j f_j \right|^p dm \\ &\leq \sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} |\alpha_j| \left(\int_{A_n \setminus A_{n+1}} |f_j|^p dm \right)^{1/p} \right)^p \\ &\leq \sum_{n=1}^{\infty} \left(|\alpha_n| \left(\int_{A_n \setminus A_{n+1}} |f_n|^p dm \right)^{1/p} + \sum_{j \neq n} |\alpha_j| \left(\int_{A_n \setminus A_{n+1}} |f_j|^p dm \right)^{1/p} \right)^p, \end{aligned}$$

where

$$(1) \quad \left(\int_{A_n \setminus A_{n+1}} |f_j|^p dm \right)^{1/p} \leq \left(\int_{A_n} |f_j|^p dm \right)^{1/p} < 4^{-n}$$

for $j < n$ by condition (i) and

$$(2) \quad \left(\int_{A_n \setminus A_{n+1}} |f_j|^p dm \right)^{1/p} \leq \left(\int_{\mathbb{T} \setminus A_j} |f_j|^p dm \right)^{1/p} < 4^{-j}$$

for $j > n$ by condition (ii). Thus by estimates (1) and (2), it always holds that

$$(3) \quad \left(\int_{A_n \setminus A_{n+1}} |f_j|^p dm \right)^{1/p} < 2^{-n-j}$$

for $j \neq n$. By using estimate (3) we get

$$\begin{aligned} \|S(\alpha)\|_p^p &\leq \sum_{n=1}^{\infty} \left(|\alpha_n| \left(\int_{A_n \setminus A_{n+1}} |f_n|^p dm \right)^{1/p} + \sum_{j \neq n} |\alpha_j| 2^{-n-j} \right)^p \\ &\leq \sum_{n=1}^{\infty} (|\alpha_n| + \|\alpha\|_{\ell^p} 2^{-n})^p \\ &\leq 2^p \left(\sum_{n=1}^{\infty} |\alpha_n|^p + \|\alpha\|_{\ell^p}^p \sum_{n=1}^{\infty} 2^{-np} \right) = 2^{p+1} \|\alpha\|_{\ell^p}^p, \end{aligned}$$

where we also used condition (iii) in the second inequality.

Let (c_k) be a subsequence of (b_n) . Then $(c_k) = (b_{n_k})$ for some sequence $0 < n_1 < n_2 < \dots$. By considering an isometry

$$J: \ell^p \rightarrow \ell^p, (\alpha_k) \mapsto (\beta_j),$$

where

$$\beta_j = \begin{cases} \alpha_k, & \text{if } j = n_k \text{ for some } k \\ 0, & \text{otherwise,} \end{cases}$$

we see that the operator $V = SJ$ is bounded. \square

For a non-compact bounded operator U on a Banach space of analytic functions, there exists a weakly (or weak-star in non-reflexive space case) convergent sequence (g_n) so that the sequence (Ug_n) of image points does not converge to zero in norm. The next result states that for a non-compact operator T_g on H^p we can find a sequence (f_k) of test functions converging weakly to zero (or in the weak-star topology for $p = 1$) so that the sequence $(T_g f_k)$ converges to a positive constant in norm. The proof is based on Theorem 2.1 of Aleman and Cima.

Proposition 3.3. *Let $g \in BMOA \setminus VMOA$ and $1 \leq p < \infty$. Then there exists a constant $c > 0$ s.t.*

$$\limsup_{|a| \rightarrow 1} \|T_g f_a\|_p = c.$$

In particular, there exists a sequence $(a_k) \subset \mathbb{D}$ s.t.

$$0 < |a_1| < |a_2| < \dots < 1$$

and $a_k \rightarrow \omega \in \mathbb{T}$ so that

$$\lim_{k \rightarrow \infty} \|T_g f_k\|_p = c.$$

Proof. It follows from Theorem 2.1 that for all $t \in (0, p/2)$ there exists a constant $C = C(p, t) > 0$ s.t.

$$(4) \quad \|T_g f_a\|_p^t \geq C \|g \circ \sigma_a - g(a)\|_t^t$$

for all $a \in \mathbb{D}$, where $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$. For each $0 < q < \infty$, it holds that

$$\text{dist}(g, VMOA) \simeq \limsup_{|a| \rightarrow 1} \|g \circ \sigma_a - g(a)\|_q,$$

where the constants of comparison depend on q , see, e.g. Lemma 3 in [8]. Thus by choosing $t = p/4$ in (4) and using Lemma 3 in [8] we get

$$\limsup_{|a| \rightarrow 1} \|T_g f_a\|_p \geq C'' \limsup_{|a| \rightarrow 1} \|g \circ \sigma_a - g(a)\|_{p/4} \simeq \text{dist}(g, VMOA) > 0,$$

since $g \in BMOA \setminus VMOA$. Thus there exists a constant $c > 0$ s.t.

$$\limsup_{|a| \rightarrow 1} \|T_g f_a\|_p = c.$$

In particular, by the compactness of $\overline{\mathbb{D}}$ there exists a sequence $(a_k) \subset \mathbb{D}$ s.t. $0 < |a_1| < |a_2| < \dots < 1$ and $a_k \rightarrow \omega \in \mathbb{T}$ so that

$$\lim_{k \rightarrow \infty} \|T_g f_k\|_p = c.$$

□

The next lemma is a generalization of Lemma 5 in [8] for $1 \leq p < \infty$.

Lemma 3.4. *Let $a \in \mathbb{D}$, $1 \leq p < \infty$, $g \in BMOA$ and*

$$f_a(z) = \frac{(1 - |a|)^{1/p}}{(1 - \bar{a}z)^{2/p}}, \quad z \in \mathbb{D}.$$

Define

$$I(a) = \left\{ e^{i\theta} : |\theta - \arg(a)| < (1 - |a|)^{\frac{1}{2(2+p)}} \right\}.$$

Then

$$\lim_{|a| \rightarrow 1} \int_{\mathbb{T} \setminus I(a)} |T_g f_a|^p dm = 0.$$

Proof. By rotation invariance, we may assume that $a \in (0, 1)$. Also, $g(0) = 0$. It holds that $|1 - ase^{i\theta}| \geq C|\theta|$ for all $0 \leq s < 1$ and $|\theta| \leq \pi$, where $C > 0$ is an absolute constant. Thus for all $0 \leq s < 1$ and $(1 - a)^{\frac{1}{2(2+p)}} \leq |\theta| \leq \pi$ we have

$$|f_a(se^{i\theta})|^p \lesssim \frac{1 - a}{|1 - ase^{i\theta}|^2} \lesssim \frac{1 - a}{|\theta|^2} \leq (1 - a)^{1 - \frac{1}{2+p}}$$

and

$$|f'_a(ase^{i\theta})|^p \lesssim \frac{1 - a}{|1 - ase^{i\theta}|^{2+p}} \lesssim \frac{1 - a}{|\theta|^{2+p}} \leq (1 - a)^{1/2}.$$

For $\zeta \in \mathbb{T} \setminus I(a)$, we obtain

$$\begin{aligned} |T_g f_a(\zeta)|^p &= \left| \int_0^1 f_a(s\zeta) g'(s\zeta) \zeta ds \right|^p \\ &\leq 2^p \left(|f_a(\zeta) g(\zeta)|^p + \left(\int_0^1 |f'_a(s\zeta) g(s\zeta)| ds \right)^p \right) \\ &\lesssim (1-a)^{1-\frac{1}{2+p}} |g(\zeta)|^p + (1-a)^{1/2} \left(\int_0^1 |g(s\zeta)| ds \right)^p. \end{aligned}$$

Since $g \in BMOA \subset \mathcal{B}$, it holds that $|g(z)| \lesssim \log \left(\frac{1}{1-|z|} \right)$ and consequently $\int_0^1 |g(s\zeta)| ds \lesssim \|g\|_*$, where $C > 0$ is an absolute constant and $\|g\|_* = \sup_{a \in \mathbb{D}} \|g \circ \sigma_a - g(a)\|_2$. Therefore

$$\int_{\mathbb{T} \setminus I(a)} |T_g f_a|^p dm \lesssim (1-a)^{1-\frac{1}{2+p}} \|g\|_p^p + (1-a)^{1/2} \|g\|_*^p \rightarrow 0$$

as $a \rightarrow 1$, where $\|g\|_p \leq \sup_{a \in \mathbb{D}} \|g \circ \sigma_a - g(a)\|_p \simeq \|g\|_*$. \square

Using Lemma 3.4, we prove the following localization result for the images $T_g f_a$, $a \in \mathbb{D}$, of the test functions f_a (cf. Lemma 3.1).

Lemma 3.5. *Let $(a_k) \subset \mathbb{D}$ be s.t. $0 < |a_1| < |a_2| < \dots < 1$ and $a_k \rightarrow \omega \in \mathbb{T}$. Define*

$$A_\varepsilon = \{e^{i\theta} : |\theta - \arg(\omega)| < \varepsilon\}$$

for each $\varepsilon > 0$ and $f_k = f_{a_k}$. Then

$$(i) \lim_{k \rightarrow \infty} \int_{\mathbb{T} \setminus A_\varepsilon} |T_g f_k|^p dm = 0 \text{ for every } \varepsilon > 0.$$

$$(ii) \text{ If } k \text{ is fixed, then } \lim_{\varepsilon \rightarrow 0} \int_{A_\varepsilon} |T_g f_k|^p dm = 0.$$

Proof. (i) Let $\varepsilon > 0$. Since $a_k \rightarrow \omega$, we have $|\arg(a_k) - \arg(\omega)| < \frac{\varepsilon}{2}$ and $(1 - |a_k|)^{\frac{1}{2(2+p)}} < \frac{\varepsilon}{2}$ for k large enough. Consequently we have

$$I(a_k) = \left\{ e^{i\theta} : |\theta - \arg(a_k)| < (1 - |a_k|)^{\frac{1}{2(2+p)}} \right\} \subset A_\varepsilon$$

for k large enough. Thus by Lemma 3.4

$$\int_{\mathbb{T} \setminus A_\varepsilon} |T_g f_k|^p dm \leq \int_{\mathbb{T} \setminus I(a_k)} |T_g f_k|^p dm \rightarrow 0$$

as $k \rightarrow \infty$.

(ii) If k is fixed, then it follows from the absolute continuity of a measure $B \mapsto \int_B |T_g f_k|^p dm$ that $\int_{A_\varepsilon} |T_g f_k|^p dm \rightarrow 0$ as $\varepsilon \rightarrow 0$. \square

As a final step before the proof of Theorem 1.1, we construct an isomorphism $U: \ell^p \rightarrow H^p$ using a non-compact T_g and test functions.

Theorem 3.6. *Let $g \in BMOA \setminus VMOA$, $1 \leq p < \infty$ and $(a_n) \subset \mathbb{D}$ be the sequence from Proposition 3.3. Then there exists a subsequence $(b_n) \subset (a_n)$ s.t. the mapping*

$$U: \ell^p \rightarrow H^p, U(\alpha) = \sum_{n=1}^{\infty} \alpha_n T_g f_n,$$

where $\alpha = (\alpha_n) \in \ell^p$ and $f_n = f_{b_n}$, is an isomorphism onto its image.

Proof. We need to show that $\|U(\alpha)\|_p \simeq \|\alpha\|_{\ell^p}$ for all $\alpha = (\alpha_n) \in \ell^p$. By Proposition 3.2 there exists a subsequence $(c_n) \subset (a_n)$ inducing a bounded operator

$$S: \ell^p \rightarrow H^p, S(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_{c_n}$$

and for any subsequence (b_n) of (c_n) the operator

$$V: \ell^p \rightarrow H^p, V(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_{b_n}$$

is bounded. Therefore the upperbound \lesssim follows from Proposition 3.2 and the boundedness of the operator T_g :

$$\begin{aligned} \left\| T_g \left(\sum_{n=1}^{\infty} \alpha_n f_{b_n} \right) \right\|_p &\leq \|T_g\|_{H^p \rightarrow H^p} \left\| \sum_{n=1}^{\infty} \alpha_n f_{b_n} \right\|_p = \|T_g\|_{H^p \rightarrow H^p} \|V(\alpha)\| \\ (5) \quad &\lesssim \|T_g\|_{H^p \rightarrow H^p} \|\alpha\|_{\ell^p}, \end{aligned}$$

where (b_n) is any subsequence of (c_n) .

Before proving the lowerbound \gtrsim , we make some preparations. Since $(c_n) \subset (a_n)$, it holds that $c_n \rightarrow \omega \in \mathbb{T}$ and there exists a constant $c > 0$ s.t. $\lim_{n \rightarrow \infty} \|T_g f_{c_n}\|_p = c$ by Proposition 3.3. For each $\varepsilon > 0$, we define a set $A_\varepsilon = \{e^{i\theta} : |\theta - \arg(\omega)| < \varepsilon\}$. Also, we define sequences (ε_n) and $(b_n) \subset (c_n)$ inductively using Proposition 3.3 and Lemma 3.5 in the following way:

We choose positive numbers ε_n and $b_n \in (c_n)$ with

$$\varepsilon_1 > \varepsilon_2 > \dots > 0$$

s.t. the following conditions hold

- (i) $\left(\int_{A_n} |T_g f_j|^p dm \right)^{1/p} < 4^{-n} \delta c, \quad j = 1, \dots, n-1;$
- (ii) $\left(\int_{\mathbb{T} \setminus A_n} |T_g f_n|^p dm \right)^{1/p} < 4^{-n} \delta c;$
- (iii) $\frac{c}{2} \leq \left(\int_{A_n} |T_g f_n|^p dm \right)^{1/p} \leq 2c$

for every $n \in \mathbb{N}$, where $A_n = A_{\varepsilon_n}$, $f_n = f_{b_n}$ and $\delta > 0$ is a constant whose value is determined later.

Now we are ready to prove the lower estimate $\|U\alpha\|_p \geq C\|\alpha\|_{\ell^p}$, where the constant $C > 0$ may depend on p .

$$\begin{aligned} \|U\alpha\|_p^p &= \int_{\mathbb{T}} \left| \sum_{j=1}^{\infty} \alpha_j T_g f_j \right|^p dm = \sum_{n=1}^{\infty} \int_{A_n \setminus A_{n+1}} \left| \sum_{j=1}^{\infty} \alpha_j T_g f_j \right|^p dm \\ &\geq \sum_{n=1}^{\infty} \left(|\alpha_n| \left(\int_{A_n \setminus A_{n+1}} |T_g f_n|^p dm \right)^{1/p} - \sum_{j \neq n} |\alpha_j| \left(\int_{A_n \setminus A_{n+1}} |T_g f_j|^p dm \right)^{1/p} \right)^p, \end{aligned}$$

where

$$\left(\int_{A_n \setminus A_{n+1}} |T_g f_j|^p dm \right)^{1/p} \leq \left(\int_{A_n} |T_g f_j|^p dm \right)^{1/p} < 4^{-n} \delta c$$

for $j < n$ by condition (i) and

$$\left(\int_{A_n \setminus A_{n+1}} |T_g f_j|^p dm \right)^{1/p} \leq \left(\int_{\mathbb{T} \setminus A_j} |T_g f_j|^p dm \right)^{1/p} < 4^{-j} \delta c$$

for $j > n$ by condition (ii). Thus it always holds that

$$\left(\int_{A_n \setminus A_{n+1}} |T_g f_j|^p dm \right)^{1/p} < 2^{-n-j} \delta c$$

for $j \neq n$. Consequently, we can estimate

$$\begin{aligned} \|U\alpha\|_p^p &\geq \sum_{n=1}^{\infty} \left(|\alpha_n| \left(\int_{A_n \setminus A_{n+1}} |T_g f_n|^p dm \right)^{1/p} - \sum_{j=1}^{\infty} |\alpha_j| 2^{-n-j} \delta c \right)^p \\ &\geq \sum_{n=1}^{\infty} \left(|\alpha_n| \left(\frac{c}{2} - 4^{-n-1} \delta c \right) - \|\alpha\|_{\ell^p} 2^{-n} \delta c \right)^p \\ &\geq \sum_{n=1}^{\infty} \left(\frac{c}{2} |\alpha_n| - \|\alpha\|_{\ell^p} (4^{-n-1} + 2^{-n}) \delta c \right)^p \\ &\geq \sum_{n=1}^{\infty} \left(\frac{c}{2} |\alpha_n| - 2^{-n+1} \delta c \|\alpha\|_{\ell^p} \right)^p \\ &\geq \sum_{n=1}^{\infty} \left(2^{-p} \left(\frac{c}{2} \right)^p |\alpha_n|^p - 2^{(-n+1)p} \delta^p c^p \|\alpha\|_{\ell^p}^p \right) \\ &= 2^{-2p} c^p \|\alpha\|_{\ell^p}^p - 2^p \delta^p c^p \left(\sum_{n=1}^{\infty} 2^{-np} \right) \|\alpha\|_{\ell^p}^p \\ &\geq (2^{-2p} - 2\delta^p) c^p \|\alpha\|_{\ell^p}^p = 2^{-2p-1} c^p \|\alpha\|_{\ell^p}^p, \end{aligned}$$

when we choose $\delta > 0$ s.t. $2^{-2p} - 2\delta^p = 2^{-2p-1}$, i.e. $\delta = 2^{-2-2/p}$. Thus the mapping U is bounded from below and by (5) it is also bounded. Therefore we have established that

$$\|U(\alpha)\|_p \simeq \|\alpha\|_{\ell^p}$$

for all $\alpha \in \ell^p$ and consequently the mapping U is an isomorphism onto its image. \square

Now we are ready to prove our main result.

Proof of Theorem 1.1. By Theorem 3.6 and Proposition 3.2, we can choose a sequence $(b_n) \subset \mathbb{D}$ that induces an isomorphism

$$U: \ell^p \rightarrow H^p, U(\alpha) = \sum_{n=1}^{\infty} \alpha_n T_g f_n$$

onto its image and a bounded operator

$$V: \ell^p \rightarrow H^p, V(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_n,$$

where $f_n = f_{b_n}$ and $\alpha = (\alpha_n) \in \ell^p$.

Define $M = \overline{\text{span}\{f_n\}}$, where the closure is taken in H^p . It is enough to show that the restriction

$$T_g|_M: M \rightarrow T_g(M)$$

is bounded from below and M is isomorphic to ℓ^p . Let $f \in M$. Then $f = \sum_{n=1}^{\infty} \alpha_n f_n$ for some $\alpha = (\alpha_n) \in \ell^p$ and it follows from the fact that U is bounded from below and the boundedness of V that

$$\begin{aligned} \|T_g f\|_p &= \left\| \sum_{n=1}^{\infty} \alpha_n T_g f_n \right\|_p = \|U(\alpha)\|_p \gtrsim \|\alpha\|_{\ell^p} \gtrsim \|V(\alpha)\|_p \\ &= \left\| \sum_{n=1}^{\infty} \alpha_n f_n \right\|_p = \|f\|_p. \end{aligned}$$

Since the operator $T_g|_M$ is also bounded, it is an isomorphism. Moreover, it holds that ℓ^p is isomorphic to $U(\ell^p) = T_g(M)$, which is isomorphic to M . Consequently the operator T_g fixes an isomorphic copy of ℓ^p , namely the closed subspace M . Hence the operator T_g is not strictly singular. \square

4. SOME COMMENTS

It follows from an idea of Leĭbov [11] that there exists isomorphic copies of the space c_0 of null sequences inside $VMOA$. Therefore the strict singularity of T_g on $BMOA$ or on $VMOA$ is equivalent to the

compactness of T_g on the same space. The sketch of the proof is the following:

First, we give a reformulation of Leĭbov's result, which is taken from [9].

Lemma 4.1 ([9, Proposition 6]). *Let (f_n) be a sequence in $VMOA$ such that $\|f_n\|_* \simeq 1$ and $\|f_n\|_2 \rightarrow 0$ as $n \rightarrow \infty$. Then there is a subsequence (f_{n_j}) which is equivalent to the natural basis of c_0 ; that is, the map $\iota: (\lambda_j) \rightarrow \sum_j \lambda_j f_{n_j}$ is an isomorphism from c_0 into $VMOA$.*

For each arc $I \subset \mathbb{T}$, we write $|I|$ to denote the length of I and define Carleson windows

$$S(I) = \{re^{it} : 1 - |I| \leq r < 1, t \in I\}$$

and their corresponding base points $u = (1 - |I|)e^{i\theta}$, where θ is the mid-point of I . We also consider “logarithmic $BMOA$ ” space

$$LMOA = \left\{ g \in H(\mathbb{D}) : \sup_{a \in \mathbb{D}} \lambda(a) \|g \circ \sigma_a - g(a)\|_2 < \infty \right\},$$

where $\lambda(a) = \log \left(\frac{2}{1-|a|} \right)$. The condition $g \in LMOA$ characterizes the boundedness of T_g on $BMOA$ and simultaneously on $VMOA$, see [14].

We consider test functions $f_n(z) = \log(1 - \bar{u}_n z)$, where $u_n \in \mathbb{D}$ is the base point of the Carleson window $S(I_n)$ and (I_n) is a sequence of arcs of \mathbb{T} s.t $I_n \rightarrow 0$. Define $h_n = f_{n+1} - f_n$. By the proof of Theorem 2 in [8], it holds that $\|h_n\|_* \simeq 1$ and $\|h_n\|_2 \rightarrow 0$, as $n \rightarrow \infty$. By Lemma 4.1, we can pick a subsequence $(h_{n_k}) \subset (h_n)$ which is equivalent to the standard basis $\{e_k\}$ of c_0 . If T_g is non-compact on $VMOA$, by passing to a subsequence if necessary, we can assume that $\|T_g h_{n_k}\|_* > c > 0$ for some constant c for all k . Since $g \in LMOA \subset BMOA$, the operator T_g is bounded on H^2 and consequently $\|T_g h_{n_k}\|_2 \rightarrow 0$, as $k \rightarrow \infty$. Now we apply Lemma 4.1 again to obtain (by passing to a subsequence, if needed) that $\{T_g h_{n_k}\}$ is equivalent to the natural basis of c_0 . Hence $T_g|_M$, where $M = \overline{\text{span}\{h_{n_k}\}}$, is an isomorphism onto its image and T_g is not strictly singular on $VMOA$ (or on $BMOA$).

Remark. In Bergman spaces A^p , $1 \leq p < \infty$, which are isomorphic to ℓ^p , see e.g. [15, Chapter 2.A, Theorem 11], the strict singularity of the operator T_g coincides with the compactness, since all strictly singular operators on ℓ^p are compact.

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